

ON THE MAXIMUM NUMBER OF POINTS IN A MAXIMAL INTERSECTING FAMILY OF FINITE SETS

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ABSTRACT. Paul Erdős and László Lovász proved in a landmark article that, for any positive integer k , up to isomorphism there are only finitely many *maximal intersecting families of k -sets* (maximal k -cliques). So they posed the problem of determining or estimating the largest number $N(k)$ of the points in such a family. They also proved by means of an example that $N(k) \geq 2k - 2 + \frac{1}{2} \binom{2k-2}{k-1}$. Much later, Zolt Tuza proved that the bound is best possible up to a multiplicative constant by showing that asymptotically $N(k)$ is at most 4 times this lower bound. In this paper we reduce the gap between the lower and upper bound by showing that asymptotically $N(k)$ is at most 3 times the Erdős-Lovász lower bound. Conjecturally, the explicit upper bound obtained in this paper is only double the lower bound.

1. INTRODUCTION

By a *family* we shall mean a family of finite sets. For a family \mathcal{F} , the members of \mathcal{F} are called its *blocks* and the elements of the blocks are called its *points*. In other words, the *point set* of \mathcal{F} , denoted by $P_{\mathcal{F}}$, is the union of all its blocks. In case \mathcal{F} is finite, we shall denote its number of points (size of the point set) by $v(\mathcal{F})$.

A family \mathcal{F} is said to be *uniform* if all its blocks have the same size. If \mathcal{F} is a uniform family we shall denote its common block size by $k(\mathcal{F})$. A *blocking set* of a family \mathcal{F} is a set A which intersects every block of \mathcal{F} . We define a *transversal* of \mathcal{F} to be a blocking set of \mathcal{F} with the smallest possible size – in case \mathcal{F} has a finite blocking set. In this case we denote by $\text{tr}(\mathcal{F})$ the common size of its transversals. If \mathcal{F} has no finite blocking set we may put $\text{tr}(\mathcal{F}) = \infty$. (Warning: Many authors use the word transversal as a synonym for blocking sets.) If $\text{tr}(\mathcal{F}) < \infty$, we denote the family of transversals of \mathcal{F} by \mathcal{F}^{\top} . Note that \mathcal{F}^{\top} is a uniform family with $k(\mathcal{F}^{\top}) = \text{tr}(\mathcal{F})$. Now we introduce:-

Definition 1.1. A family \mathcal{F} is said to be a *maximal intersecting family* (in short *MIF*) if $\text{tr}(\mathcal{F}) < \infty$ and $\mathcal{F} = \mathcal{F}^{\top}$. We use $MIF(k)$ as a generic name for *MIF*'s with $k(\mathcal{F}) = k$.

We say that a family \mathcal{F} is an *intersecting family* if any two blocks of \mathcal{F} have non empty intersection. Clearly any $MIF(k)$ is an intersecting family. Indeed, the $MIF(k)$'s are characterized among all k -uniform intersecting families as those families which are maximal with respect to the property of being intersecting. Thus, an intersecting family \mathcal{F} of k -sets is a $MIF(k)$ if and only if there is no k -set outside \mathcal{F} (anywhere in the universe of all sets!) which is a blocking set of \mathcal{F} . In the hypergraph literature these are known as the *maximal k -cliques*.

In [2] Erdős and Lovász proved the surprising result that any $MIF(k)$ is finite; indeed it has at most k^k blocks. In Theorem 2.1 we point out that, more generally, for any k -uniform family \mathcal{F} with finite transversal size $\text{tr}(\mathcal{F}) = t$, the family \mathcal{F}^{\top} is finite. Indeed, $\#(\mathcal{F}^{\top}) \leq k^t$.

In view of the result of Erdős and Lovász quoted above, we see that, for any fixed $k \geq 1$, there are only finitely many $MIF(k)$'s, up to isomorphism. This led Erdős and Lovász to ask for the determination of the maximum possible number $N(k)$ of points among all $MIF(k)$'s. Thus

$$N(k) := \max \{v(\mathcal{F}) : \mathcal{F} \text{ is a } MIF(k)\}.$$

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By means of an explicit construction in [2] it was proved that

$$N(k) \geq 2k - 2 + \frac{1}{2} \binom{2k-2}{k-1}. \quad (1)$$

Note that the lower bound in (1) is asymptotically $\frac{1}{2} \binom{2k-2}{k-1}$. In 1985, Tuza proved that, up to a multiplicative constant this is best possible. In order to explain Tuza's contribution, we recall

Definition 1.2. An *intersecting set pair system* (in short *ISP*) is a collection $\{(A_i, B_i) : 1 \leq i \leq n\}$ of pairs of finite sets with the property that, for $1 \leq i, j \leq n$, $A_i \cap B_j = \emptyset$ if and only if $i = j$. Clearly, in such a system, the sets A_i (as well as the sets B_i) are distinct. The set $\bigcup_{i=1}^n (A_i \sqcup B_i)$ is called the *point set* of the *ISP*. We denote by $v(\mathbb{I})$ the number of points of an *ISP* \mathbb{I} . If in \mathbb{I} , $\#(A_i) = k$ and $\#(B_i) = t$ for $1 \leq i \leq n$, then we say that \mathbb{I} is an *ISP* with parameter (k, t) . We use $ISP(k, t)$ as a generic name for an *ISP* with parameter (k, t) .

In [1], Bollobás proved the following inequality for arbitrary *ISP*'s. If $\{(A_i, B_i) : 1 \leq i \leq n\}$ is an *ISP* then

$$\sum_{i=1}^n \frac{1}{\binom{\#(A_i) + \#(B_i)}{\#(A_i)}} \leq 1. \quad (2)$$

In particular, for any $ISP(k, t)$ consisting of n pairs, we have Bollobás inequality

$$n \leq \binom{k+t}{k}. \quad (3)$$

This inequality shows that, for any two positive integers k and t , there are only finitely many $ISP(k, t)$, up to isomorphism. This raises the question of determining or estimating the number

$$n(k, t) := \max \{v(\mathbb{I}) : \mathbb{I} \text{ is an } ISP(k, t)\}$$

Notice that we have $n(k, t) = n(t, k)$.

In Theorem 6(a) of [4], Tuza used an extremely elegant argument to deduce the following bound from Bollobás inequality (2). (The sum here is a simplification of the sum given by Tuza.)

$$\text{For } k \geq t, \quad n(k, t) \leq \binom{k+t}{t+1} - \binom{2t-1}{t+1} + \frac{3}{2} \sum_{i=1}^{t-1} \binom{2i}{i}. \quad (4)$$

A family \mathcal{F} is 1-critical according to Tuza if for any $x \in B \in \mathcal{F}$, there is a $B' \in \mathcal{F}$ such that $B \cap B' = \{x\}$. Notice that any $MIF(k)$ is 1-critical (else $B \setminus \{x\}$ would be a blocking set). In Corollary 12 of [4], Tuza observes that $n(k, k-1)$ is an upper bound on the number of points in any k -uniform 1-critical family. In particular this applies to $MIF(k)$'s. So we have

$$N(k) \leq n(k, k-1). \quad (5)$$

Substituting $t = k-1$ in (4) we therefore get

$$N(k) \leq \frac{3}{2} \sum_{i=1}^{k-1} \binom{2i}{i} \sim 2 \binom{2k-2}{k-1}. \quad (6)$$

where the asymptotics is determined by Stirling's asymptotic formula for factorials and summation by parts. Thus, as $k \rightarrow \infty$, Tuza's upper bound is 4 times the lower bound given by Erdős and Lovász.

The main object of this paper is to improve the estimate (6) on $N(k)$. The method adopted here is inspired by Tuza [4]. We introduce the problem of finding or estimating the number

$$N^\top(k, t) := \max \{v(\mathcal{F}^\top) : \mathcal{F} \text{ is a uniform family with } k(\mathcal{F}) = k \text{ and } \text{tr}(\mathcal{F}) = t\}.$$

(Note that we are trying to maximize the size of the point set of the family of transversals of \mathcal{F} , which in general is a subset of the point set of \mathcal{F} .) This number is finite in view of Theorem 2.1 below. In Theorem 2.4 we prove:

$$N^\top(k, t) \leq n(k, t-1). \quad (7)$$

In Theorem 2.6, we show that, given any $MIF(k)$ \mathcal{F} such that \mathcal{F} has a pair $\{\alpha, \beta\}$ of points not contained in any block of \mathcal{F} , one can construct another $MIF(k)$, denoted $\mathcal{F}[\beta \mapsto \alpha]$, with one less point. Among the blocks of the later $MIF(k)$ are included the sets $\{\alpha\} \sqcup (B \setminus \{\beta\})$, $\beta \in B \in \mathcal{F}$; hence the name. One might imagine that a method to reduce the number of points in a $MIF(k)$ can't have much to do with the problem of estimating the number $N(k)$ of the largest possible number of points in a $MIF(k)$. However, our final result (Theorem 2.7) is a new upper bound on $N(k)$ obtained by combining Theorem 2.4 and Theorem 2.6 with Bollobás inequality (3). Here we prove

$$N(k) \leq \frac{1}{2} \binom{2k-2}{k-1} + n(k, k-2). \quad (8)$$

In view of Tuza's inequality (4), this yields the bound

$$N(k) \leq \frac{3}{2} \sum_{i=1}^{k-1} \binom{2i}{i} - \frac{1}{2} \binom{2k-2}{k-1} \sim \frac{3}{2} \binom{2k-2}{k-1}. \quad (9)$$

Thus as $k \rightarrow \infty$, $N(k)$ is at most 3 times the lower bound (1) of Erdős and Lovász.

In [3], Hanson and Toft proved that, actually, $N(k) = 2k - 2 + \frac{1}{2} \binom{2k-2}{k-1}$ for $2 \leq k \leq 4$. In conjunction with Tuza's bound (6) and its improvement (9), this result leads us to pose:

Conjecture. For $k \geq 2$, $N(k) = 2k - 2 + \frac{1}{2} \binom{2k-2}{k-1}$.

It may be noted that Tuza constructed ([4, Construction 11]) a k -uniform 1-critical family with $2k - 4 + 2 \binom{2k-4}{k-2}$ points. This number is larger than $2k - 2 + \frac{1}{2} \binom{2k-2}{k-1}$ for $k \geq 5$. However, as already noted, the class of 1-critical uniform families is larger than that of MIF 's. Indeed, the families constructed by Tuza are not MIF 's. So this construction does not disprove the above conjecture.

Recall that the *chromatic number* of a family is the smallest number of colours using which the points may be coloured so that no monochromatic block occurs. It is trivial to see that any uniform intersecting family (clique) \mathcal{F} has chromatic number at most 3. (Choose $x \in B \in \mathcal{F}$. Assign the first colour to x , second colour to the other points of B and the third colour to the remaining points.) Thus any such family is either 2-chromatic or 3-chromatic. The article [2] was mainly concerned with k -uniform 3-chromatic intersecting families. This is a subclass of the class of $MIF(k)$'s. Indeed, a k -uniform intersecting family is 3-chromatic if and only if its blocks are the only minimal (as opposed to just minimum sized) blocking sets. (The *finite projective planes* of order $q \geq 3$ are examples of 2-chromatic $MIF(q+1)$.) So of course, the lower bound (1) holds for all $MIF(k)$'s.

Finally, we note that in [4], Tuza has made a precise conjecture on the numbers $n(k, t)$:

Conjecture (Tuza). For $k \geq t + 2$,

$$n(k, t) = \left\lceil \frac{k}{t+1} \right\rceil \left(\left\lfloor \frac{kt}{t+1} \right\rfloor + t \right) + \left\lfloor \frac{kt}{t+1} \right\rfloor + t$$

If this is correct, our bound (8) becomes

$$N(k) \leq \frac{1}{2} \binom{2k-2}{k-1} + 2 \binom{2k-4}{k-2} + 2k - 4 \sim \binom{2k-2}{k-1},$$

which is asymptotically double the conjectured value.

2. PROOFS

Recall that, for any finite family \mathcal{F} , $v(\mathcal{F})$ is its number of points and $P_{\mathcal{F}}$ is its point set. If \mathcal{F} is uniform, $k(\mathcal{F})$ is its common block size. \mathcal{F}^{\top} is the family of transversals of \mathcal{F} and $\text{tr}(\mathcal{F})$ is the common size of the transversals. $N(k)$ is the maximum of $v(\mathcal{F})$ over all $MIF(k)$ \mathcal{F} . $N^{\top}(k, t)$ is the maximum of $v(\mathcal{F}^{\top})$ over all \mathcal{F} with $k(\mathcal{F}) = k$ and $\text{tr}(\mathcal{F}) = t$. Also $n(k, t)$ is the maximum of $v(\mathbb{I})$ over all $ISP(k, t)$ \mathbb{I} .

Theorem 2.1. If $k(\mathcal{F}) = k$ and $\text{tr}(\mathcal{F}) = t$ then $\#(\mathcal{F}^{\top}) \leq k^t$.

Proof : This is the $s = 0$ case of the following.

Claim : For $0 \leq s \leq t$, any set of s points of \mathcal{F} are together contained in at most k^{t-s} transversals of \mathcal{F} .

Proof of the Claim : We prove this claim by backward induction on s . It is trivial for $s = t$. So suppose the claim holds for some s , with $1 \leq s \leq t$. Take any set A of $s - 1$ points. Since $\text{tr}(\mathcal{F}) = t > \#(A)$, A is not a blocking set of \mathcal{F} . So there is a block $B \in \mathcal{F}$ disjoint from A . Therefore each transversal containing A contains at least one of the k sets $A \sqcup \{x\}$, $x \in B$. By induction hypothesis, $A \sqcup \{x\}$ is contained in at most k^{t-s} transversals for each $x \in B$. Therefore A is contained in at most $k \cdot k^{t-s} = k^{t-(s-1)}$ transversals. This completes the induction. \square

Corollary 2.2. *Let k, t be positive integers. Then up to isomorphism, there are only finitely many families \mathcal{G} with $k(\mathcal{G}) = t$ such that \mathcal{G} is isomorphic to \mathcal{F}^\top for some uniform family \mathcal{F} with $k(\mathcal{F}) = k$.*

Proof : By Theorem 2.1, any such \mathcal{G} has at most k^t blocks; hence it has at most $t \cdot k^t$ points. Therefore up to isomorphism, we may assume that all such families \mathcal{G} are contained in the power set of a fixed set of size tk^t . So there are only finitely many \mathcal{G} 's. \square

This corollary shows that $N(k)$ and $N^\top(k, t)$ are both finite.

Construction 2.3. Let $2 \leq t \leq k - 1$ and S be a set of $k + t - 2$ symbols. Let $\binom{S}{i}$ denote the family consisting of all i -subsets of S . Take a new symbol x_A (from outside S) for each $A \in \binom{S}{k-1}$. Let

$$\mathcal{F} = \binom{S}{k} \sqcup \left\{ \{x_A\} \sqcup A : A \in \binom{S}{k-1} \right\}.$$

It is easy to verify that $\text{tr}(\mathcal{F}) = t$ and

$$\mathcal{F}^\top = \binom{S}{t} \sqcup \left\{ \{x_A\} \sqcup (S \setminus A) : A \in \binom{S}{k-1} \right\}.$$

Theorem 2.4. *For $2 \leq t \leq k - 1$,*

$$k + t - 2 + \binom{k+t-2}{t-1} \leq N^\top(k, t) \leq n(k, t - 1).$$

Proof : Construction 2.3 yields a k -uniform family \mathcal{F} such that $\text{tr}(\mathcal{F}) = t$ and \mathcal{F}^\top has $k + t - 2 + \binom{k+t-2}{t-1}$ points. Hence we get the lower bound.

Let \mathcal{F} be a k -uniform family with $\text{tr}(\mathcal{F}) = t$. We need to show that $v(\mathcal{F}^\top) \leq n(k, t - 1)$. Let

$$\mathcal{E} = \{B_i : 1 \leq i \leq n\}$$

be a minimal subfamily of \mathcal{F} such that $\text{tr}(\mathcal{E}) = t$. Then, for $1 \leq i \leq n$, $\mathcal{E}_i := \mathcal{E} \setminus \{B_i\}$ has $\text{tr}(\mathcal{E}_i) = t - 1$. Choose a transversal T_i of \mathcal{E}_i , where $1 \leq i \leq n$. Since $\text{tr}(\mathcal{E}) = t$, it follows that $T_i \cap B_i = \emptyset$. Thus

$$\mathbb{I} = \{(B_i, T_i) : 1 \leq i \leq n\}$$

is an $ISP(k, t - 1)$. Therefore, to complete the proof, it suffices to show that each point x of \mathcal{F}^\top is a point of \mathbb{I} . Choose a transversal T of \mathcal{F} such that $x \in T$. Then T intersects all the B_i 's. If x was not a point of \mathcal{E} then $T \setminus \{x\}$ would be a blocking set of \mathcal{E} , of size $t - 1$, contradicting the choice of \mathcal{E} . So x is a point of \mathcal{E} and hence of \mathbb{I} . \square

Since, clearly, $N(k) \leq N^\top(k, k)$, Theorem 2.4 includes Tuza's upper bound (5) on $N(k)$.

Construction 2.5. Let \mathcal{F} be a $MIF(k)$ and suppose $\alpha \neq \beta$ are two points of \mathcal{F} such that no block of \mathcal{F} contains $\{\alpha, \beta\}$. Let $\mathcal{G} := \{B \in \mathcal{F} : \alpha \notin B, \beta \notin B\}$. Put

$$\mathcal{F}[\beta \mapsto \alpha] := \mathcal{G} \sqcup \{T \sqcup \{\alpha\} : T \in \mathcal{G}^\top\}.$$

Theorem 2.6. *Let α, β be two points of a $MIF(k)$ \mathcal{F} such that no block of \mathcal{F} contains both α and β . Then the family $\mathcal{F}[\beta \mapsto \alpha]$ (given by Construction 2.5) is a $MIF(k)$ with point set $P_{\mathcal{F}} \setminus \{\beta\}$.*

Proof : Let \mathcal{G} be as in Construction 2.5. If T is transversal of \mathcal{G} with $\#(T) \leq k - 2$, then $T \sqcup \{\alpha, \beta\}$ is a blocking set of \mathcal{F} of size at most k . Since \mathcal{F} is a $MIF(k)$, it follows that $T \sqcup \{\alpha, \beta\}$ is a block of \mathcal{F} . This is a contradiction since no block of \mathcal{F} contains both α and β . Thus $\text{tr}(\mathcal{G}) \geq k - 1$. Since, for $\beta \in B \in \mathcal{F}$, $B \setminus \{\beta\}$ is a blocking set of \mathcal{G} , it follows that $\text{tr}(\mathcal{G}) = k - 1$. Thus $\hat{\mathcal{F}} := \mathcal{F}[\beta \mapsto \alpha]$ is uniform with $k(\hat{\mathcal{F}}) = k$. This argument also shows that if $\beta \notin B \in \mathcal{F}$, then B is a block of $\hat{\mathcal{F}}$. Also if $\beta \in B \in \mathcal{F}$, then $\{\alpha\} \sqcup (B \setminus \{\beta\})$ is a block of $\hat{\mathcal{F}}$. We have the following.

Claim : For each $T \in \mathcal{G}^\top$ there exists $T' \in \mathcal{G}^\top$ such that $T \cap T' = \emptyset$.

Proof of the Claim : Suppose the claim is false. Then there exists $T \in \mathcal{G}^\top$ such that T is a blocking set of \mathcal{G}^\top . So T is a blocking set of $\mathcal{G} \sqcup \mathcal{G}^\top$, and hence of \mathcal{F} . This means $\text{tr}(\mathcal{F}) \leq \#(T) = k - 1$. Contradiction.

Let C be a blocking set of $\hat{\mathcal{F}}$. Then in particular it is a blocking set of \mathcal{G} . Since $\text{tr}(\mathcal{G}) = k - 1$ it follows that $\#(C) \geq k - 1$. If $\#(C) = k - 1$ then $C \in \mathcal{G}^\top$, so that $\alpha \notin C$. By the above claim there exists a $T \in \mathcal{G}^\top$ such that $T \cap C = \emptyset$. Hence C is disjoint from $T \sqcup \{\alpha\} \in \hat{\mathcal{F}}$. Contradiction. Hence $\#(C) \geq k$. Therefore $\text{tr}(\hat{\mathcal{F}}) = k$. Since \mathcal{F} is an intersecting family, the construction of $\hat{\mathcal{F}}$ shows that $\hat{\mathcal{F}}$ is an intersecting family. Consequently $\hat{\mathcal{F}} \subseteq (\hat{\mathcal{F}})^\top$. If T is a transversal of $\hat{\mathcal{F}}$ and $\alpha \in T$, then $T \setminus \{\alpha\}$ is a transversal of \mathcal{G} , so that $T = (T \setminus \{\alpha\}) \sqcup \{\alpha\} \in \hat{\mathcal{F}}$. If T is a transversal of $\hat{\mathcal{F}}$ and $\alpha \notin T$ then (as all the blocks of \mathcal{F} with $\beta \notin B$ are blocks of $\hat{\mathcal{F}}$ and for $\beta \in B \in \mathcal{F}$, $(B \setminus \{\beta\}) \sqcup \{\alpha\}$ is a block of $\hat{\mathcal{F}}$) T is a transversal of \mathcal{F} . Hence $T \in \mathcal{F}$ and $\beta, \alpha \notin T$, so that $T \in \mathcal{G} \subseteq \hat{\mathcal{F}}$. Thus $(\hat{\mathcal{F}})^\top \subseteq \hat{\mathcal{F}}$, so that $\hat{\mathcal{F}}$ is a $MIF(k)$.

Clearly the point set of $\hat{\mathcal{F}}$ is contained in $P_{\mathcal{F}} \setminus \{\beta\}$. Take any $\gamma \in P_{\mathcal{F}} \setminus \{\beta\}$. Take a block B of \mathcal{F} such that $\gamma \in B$. If $\beta \notin B$ then we have $\gamma \in B \in \hat{\mathcal{F}}$ and hence γ is a point of $\hat{\mathcal{F}}$. If $\beta \in B$, then – as $\#(B) = k = \text{tr}(\mathcal{F})$, there is a block B' of \mathcal{F} such that $B \cap B' = \{\gamma\}$. Then $\gamma \in B' \in \hat{\mathcal{F}}$, hence again γ is a point of $\hat{\mathcal{F}}$. Thus the point set of $\hat{\mathcal{F}}$ is $P_{\mathcal{F}} \setminus \{\beta\}$. \square

Theorem 2.7. For $k \geq 2$,

$$N(k) \leq \frac{1}{2} \binom{2k-2}{k-1} + n(k, k-2).$$

Proof : Let \mathcal{F} be a $MIF(k)$. We need to show that $v(\mathcal{F}) \leq \frac{1}{2} \binom{2k-2}{k-1} + n(k, k-2)$. Fix a point α of \mathcal{F} . We inductively define two finite sequences: a sequence $\{\beta_n : 0 \leq n \leq N-1\}$ of distinct points of \mathcal{F} and a sequence $\{\mathcal{F}_n : 1 \leq n \leq N\}$ of $MIF(k)$'s. Define $\beta_0 = \alpha$, $\mathcal{F}_1 = \mathcal{F}$. Suppose we have already defined β_m for $0 \leq m \leq n-1$, and \mathcal{F}_m for $1 \leq m \leq n$. If for each point β of \mathcal{F}_n there is a block of \mathcal{F}_n containing both α and β , then put $n = N$ and terminate the construction. Otherwise, choose a point β_n of \mathcal{F}_n such that no block of \mathcal{F}_n contains both α and β_n and construct $\mathcal{F}_{n+1} := \mathcal{F}_n[\beta_n \mapsto \alpha]$. By construction and Theorem 2.6, for $n \geq 1$ each \mathcal{F}_{n+1} is a $MIF(k)$ with $P_{\mathcal{F}_{n+1}} = P_{\mathcal{F}_n} \setminus \{\beta_n\}$.

Notice that this construction must end in finitely many steps, since by Theorem 2.1, $\mathcal{F}_1 = \mathcal{F}$ is finite. Since induction has terminated at the N -th step, \mathcal{F}_N has the property that for each point β of \mathcal{F}_N there is a block of \mathcal{F}_N containing both α and β . Put

$$\mathcal{G} = \{B \in \mathcal{F}_N : \alpha \notin B\}.$$

For $\alpha \in B \in \mathcal{F}_N$, $B \setminus \{\alpha\}$ is a blocking set of \mathcal{G} of size $k - 1$. So $\text{tr}(\mathcal{G}) \leq k - 1$. If T is a transversal of \mathcal{G} with $\#(T) \leq k - 1$, then $T \sqcup \{\alpha\}$ is a blocking set of \mathcal{F}_N with size at most k . Since \mathcal{F}_N is a $MIF(k)$, it follows that $T \sqcup \{\alpha\}$ is a block of \mathcal{F}_N . Thus $\text{tr}(\mathcal{G}) = k - 1$ and $\mathcal{G}^\top = \{B \setminus \{\alpha\} : \alpha \in B \in \mathcal{F}_N\}$. Thus $P_{\mathcal{G}} = P_{\mathcal{G}^\top} = P_{\mathcal{F}} \setminus \{\beta_n : 0 \leq n \leq N-1\}$. Therefore, by Theorem 2.4,

$$v(\mathcal{F}) = N + v(\mathcal{G}^\top) \leq N + N^\top(k, k-1) \leq N + n(k, k-2). \quad (10)$$

Choose two blocks B_0, B'_0 of $\mathcal{F} = \mathcal{F}_1$ such that $B_0 \cap B'_0 = \{\beta_0\}$. Also, for $1 \leq n \leq N-1$, choose two blocks B_n, B'_n of \mathcal{F}_n such that $B_n \cap B'_n = \{\beta_n\}$. (As already remarked, any point of a $MIF(k)$ lies in such a pair of blocks.) Put $T_n = B_n \setminus \{\beta_n\}$, $T'_n = B'_n \setminus \{\beta_n\}$. Thus $T_n \cap T'_n = \emptyset$ for $0 \leq n \leq N-1$.

Claim : For $0 \leq m < n \leq N-1$, $T_m \sqcup \{\alpha\}$ and $T'_m \sqcup \{\alpha\}$ are blocks of \mathcal{F}_n .

Proof of the Claim : This claim may be proved by finite induction on n .

If $n = m + 1$, then $\mathcal{F}_n = \mathcal{F}_m[\beta_m \mapsto \alpha]$ and $T_m \sqcup \{\beta_m\}, T'_m \sqcup \{\beta_m\} \in \mathcal{F}_m$ implies $T_m \sqcup \{\alpha\}, T'_m \sqcup \{\alpha\} \in \mathcal{F}_{m+1} = \mathcal{F}_n$. If $m < n \leq N - 1$, and the claim is correct for this value of n , then $T_m \sqcup \{\alpha\}, T'_m \sqcup \{\alpha\} \in \mathcal{F}_n$ and $\mathcal{F}_{n+1} = \mathcal{F}_n[\beta_n \mapsto \alpha]$ implies $T_m \sqcup \{\alpha\}, T'_m \sqcup \{\alpha\} \in \mathcal{F}_{n+1}$.

Now for $0 \leq m < n \leq N - 1$, $T_m \sqcup \{\alpha\}, T'_m \sqcup \{\alpha\}, T_n \sqcup \{\beta_n\}$ and $T'_n \sqcup \{\beta_n\}$ are blocks of the intersecting family \mathcal{F}_n . Therefore these four sets intersect pairwise. Since $\beta_n \neq \alpha$ and α & β_n are never together in a block of \mathcal{F}_n , it follows that $T_m \cap T_n \neq \emptyset, T'_m \cap T_n \neq \emptyset, T_m \cap T'_n \neq \emptyset$ and $T'_m \cap T'_n \neq \emptyset$ for $0 \leq m < n \leq N - 1$. Therefore,

$$\mathbb{I} := \left\{ (T_n, T'_n) : 0 \leq n \leq N - 1 \right\} \sqcup \left\{ (T'_n, T_n) : 0 \leq n \leq N - 1 \right\}$$

is an $ISP(k - 1, k - 1)$ containing $2N$ pairs. Therefore by Bollobás inequality (3), we get

$$N \leq \frac{1}{2} \binom{2k - 2}{k - 1}. \quad (11)$$

From (10) and (11), we conclude that $v(\mathcal{F}) \leq \frac{1}{2} \binom{2k - 2}{k - 1} + n(k, k - 2)$. \square

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